

COMPUTING QUASIDEGREES OF A -GRADED MODULES

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ABSTRACT. We describe the main functions of the **Macaulay2** package **Quasidegrees**. The purpose of this package is to compute the quasidegree set of a finitely generated \mathbb{Z}^d -graded module presented as the cokernel of a monomial matrix. We provide examples with motivation coming from A -hypergeometric systems.

1. INTRODUCTION

Throughout $R = \mathbb{k}[x_1, \dots, x_n]$ is a \mathbb{Z}^d -graded polynomial ring over a field \mathbb{k} and $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$ denotes the homogeneous maximal ideal in R . Let $M = \bigoplus_{\beta \in \mathbb{Z}^d} M_\beta$ be a \mathbb{Z}^d -graded R -module. The *true degree set* of M is

$$\text{tdeg}(M) = \{\beta \in \mathbb{Z}^d \mid M_\beta \neq 0\}.$$

The *quasidegree set* of M , denoted $\text{qdeg}(M)$, is the Zariski closure in \mathbb{C}^d of $\text{tdeg}(M)$.

The purpose of the **Macaulay2**[GS] package **Quasidegrees**[Bar] is to compute the quasidegree set of a finitely generated \mathbb{Z}^d -graded module presented as the cokernel of a monomial matrix. By a monomial matrix, we mean a matrix where each entry is either zero or a monomial in R . The initial motivation for **Quasidegrees** was to compute the quasidegree sets of certain local cohomology modules supported at \mathfrak{m} of \mathbb{Z}^d -graded R -modules so there are some methods in the package specific to local cohomology. Recall that the *i th local cohomology module* of M with support at the ideal $I \subset R$ is the *i th right derived functor* of the left exact I -torsion functor

$$\Gamma_I(M) = \{m \in M \mid I^t m = 0 \text{ for some } t \in \mathbb{N}\}$$

on the category of R -modules.

By the vanishing theorems of local cohomology [Eis95], the quasidegree sets of the local cohomology modules supported at \mathfrak{m} of M can be seen as measuring how far the module is from being Cohen-Macaulay. From the A -hypergeometric systems point of view, the quasidegree set of the non-top local cohomology modules supported at \mathfrak{m} of R/I_A , where I_A is the toric ideal associated to A in R , determine the parameters β where the A -hypergeometric system $H_A(\beta)$ has rank higher than expected (see Section 3).

2. QUASIDEGREES

The main function of `Quasidegrees` is `quasidegrees`, which computes the quasidegree set of a module that is presented by a monomial matrix.

We use the idea of standard pairs of monomial ideals to compute the quasidegree set of a \mathbb{Z}^d -graded R -module. Given a monomial $x^{\mathbf{u}}$ and a subset $Z \subset \{x_1, \dots, x_n\}$, the pair $(x^{\mathbf{u}}, Z)$ indexes the monomials $x^{\mathbf{u}} \cdot x^{\mathbf{v}}$ where $\text{supp}(x^{\mathbf{v}}) \subset Z$. A *standard pair* of a monomial ideal $I \subset R$ is a pair $(x^{\mathbf{u}}, Z)$ satisfying:

- (1) $\text{supp}(x^{\mathbf{u}}) \cap Z = \emptyset$,
- (2) all of the monomials indexed by $(x^{\mathbf{u}}, Z)$ are outside of I ,
- (3) $(x^{\mathbf{u}}, Z)$ is maximal in the sense that $(x^{\mathbf{u}}, Z) \not\subseteq (x^{\mathbf{v}}, Y)$ for any other pair $(x^{\mathbf{v}}, Y)$ satisfying the first two conditions.

To compute the quasidegree set of M we first find a monomial presentation of M so that M is the cokernel of a monomial matrix ϕ . We then compute the standard pairs of the ideals generated by the rows of ϕ and to each standard pair we associate the degrees of the corresponding variables. The following algorithm is implemented in `Quasidegrees`. The input is an R -module presented by a monomial matrix $\phi : R^s \rightarrow R^t$. As in `Macaulay2`, we write the degree of the k th factor of R^t next to the k th row of the matrix ϕ .

Algorithm 1 Compute `qdeg`(M)

Input: R -module M presented by monomial matrix

$$\phi = \alpha_i [c_{j,k} \mathbf{x}^{\mathbf{u}_{j,k}}] : R^s \rightarrow R^t$$

Output: `qdeg`(M)

$$Q = \emptyset$$

for $1 \leq k \leq t$ **do**

$$SP = \{\text{standard pairs of } \langle c_{k,1} \mathbf{x}^{\mathbf{u}_{k,1}}, c_{k,2} \mathbf{x}^{\mathbf{u}_{k,2}}, \dots, c_{k,s} \mathbf{x}^{\mathbf{u}_{k,s}} \rangle\}$$

$$Q = Q \cup \{\deg(\mathbf{x}^{\mathbf{u}}) + \alpha_k + \sum_{x_i \in F} \mathbb{C} \cdot \deg(x_i) \mid (\mathbf{x}^{\mathbf{u}}, Z) \in SP\}$$

end for

return Q

In the implementation of Algorithm 1 in `Macaulay2`, we represent the output as a list of pairs (\mathbf{u}, Z) with $\mathbf{u} \in \mathbb{Q}^d$ and $Z \subset \mathbb{Q}^d$ where the pair (\mathbf{u}, Z) represents the plane

$$\mathbf{u} + \sum_{\mathbf{v} \in Z} \mathbb{C} \cdot \mathbf{v}.$$

The union of these planes over all such pairs in the output is the quasidegree set of M .

The following is an example of `Quasidegrees` computing the quasidegree set of an R -module:

```
i1 : R=QQ[x,y,z]
o1 = R
```

```

o1 : PolynomialRing
i2 : I=ideal(x*y,y*z)
o2 = ideal (x*y, y*z)
o2 : Ideal of R
i3 : M=R^1/I
o3 = cokernel | xy yz |
                                     1
o3 : R-module, quotient of R
i4 : Q = quasidegrees M
o4 = {{0, {| 1 |}}}, {0, {| 1 |, | 1 |}}}}
o4 : List

```

The above example displays a caveat of **quasidegrees** in that there may be some redundancies in the output. By a redundancy, we mean when one plane in the output is contained in another. The redundancy above is clear:

$$\text{qdeg}(\mathbb{k}[x, y, z]/\langle xy, yz \rangle) = \mathbb{C} = \{z_1 + z_2 \in \mathbb{C} \mid z_1, z_2 \in \mathbb{C}\}.$$

The function **removeRedundancy** gets rid of redundancies in the list of planes:

```

i5 : removeRedundancy Q
o5 = {{0, {| 1 |, | 1 |}}}}
o5 : List

```

3. QUASIDEGREES AND HYPERGEOMETRIC SYSTEMS

In this section, we discuss the motivation for **Quasidegrees** and the methods in **Quasidegrees** that aid us in our studies. Let $A = [a_1 \ a_2 \ \cdots \ a_n]$ be an integer $(d \times n)$ -matrix with $\mathbb{Z}A = \mathbb{Z}^d$ and such that the cone over its columns is pointed. There is a natural \mathbb{Z}^d -grading of R by the columns of A given by $\deg(x_j) = a_j$, the j th column of A . A module that is homogeneous with respect to this grading is said to be A -graded. By the assumptions on A , R is positively graded by A , that is, the only polynomials of degree 0 are the constants. Given such a matrix A and a polynomial ring R in n variables, the method **toAgradedRing** gives R an A -grading. For example, let $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -2 \end{pmatrix}$. We make the A -graded polynomial ring $\mathbb{Q}[x_1, x_2, x_3, x_4, x_5]$:

```

i6 : A=matrix{{1,1,1,1,1},{0,0,1,1,0},{0,1,1,0,-2}}
o6 = | 1 1 1 1 1 |
      | 0 0 1 1 0 |
      | 0 1 1 0 -2 |
              3          5
o6 : Matrix ZZ <--- ZZ
i7 : R=QQ[x_1..x_5]
o7 = R
o7 : PolynomialRing
i8 : R=toAgradedRing(A,R)

```

```

o8 = R
o8 : PolynomialRing
i9 : describe R
o9 = QQ[x1, x2, x3, x4, x5, Degrees => {{1}, {1}, {1}, {1},
                                         {0} {0} {1} {1},
                                         {0} {1} {1} {0}}
-----
{1 }}, Heft => {1, 2:0}, MonomialOrder =>
{0 }
{-2}
-----
{MonomialSize => 32}, DegreeRank => 3]
{GRevLex => {5:1} }
{Position => Up    }

```

The *toric ideal associated to A in R* is the binomial ideal

$$I_A = \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} : A\mathbf{u} = A\mathbf{v} \rangle.$$

The method `toricIdeal` computes the toric ideal associated to A in the ring R . We continue with the A and R from the above example and compute the toric ideal I_A associated to A in R :

```

i10 : I=toricIdeal(A,R)
o10 = ideal (x12x32 - x22x42, x12x32 - x22x42, x12x32 - x22x42, x13 -
              1 3      2 4      1 4      3 5      1 4      2 3 5      1
              -----
              2
              x x )
              2 5

o10 : Ideal of R

```

We now introduce A -hypergeometric systems. Given a matrix $A \in \mathbb{Z}^{d \times n}$ as above and a $\beta \in \mathbb{C}^d$, the A -hypergeometric system with parameter $\beta \in \mathbb{C}^d$ [SST00], denoted $H_A(\beta)$, is the system of partial differential equations:

$$\frac{\partial^{|\mathbf{v}|}}{\partial \mathbf{x}^{\mathbf{v}}} \phi(\mathbf{x}) = \frac{\partial^{|\mathbf{u}|}}{\partial \mathbf{x}^{\mathbf{u}}} \phi(\mathbf{x}) \text{ for all } \mathbf{u}, \mathbf{v}, A\mathbf{u} = A\mathbf{v}$$

$$\sum_{j=1}^n a_{ij} x_j \frac{\partial}{\partial x_j} \phi(\mathbf{x}) = \beta_i \phi(\mathbf{x}), \text{ for } i = 1, \dots, d.$$

Such systems are sometimes called *GKZ-hypergeometric systems*. The function `gkz` in the `Macaulay2` package `Dmodules` computes this system as

an ideal in the Weyl algebra. The *rank* of $H_A(\beta)$ is

$$\text{rank}(H_A(\beta)) = \dim_{\mathbb{C}} \left\{ \begin{array}{l} \text{germs of holomorphic solutions of } H_A(\beta) \\ \text{near a generic nonsingular point} \end{array} \right\}.$$

The function `holonomicRank` in `Dmodules` computes the rank of an A -hypergeometric system. In general, rank is not a constant function of β . Denote $\text{vol}(A)$ to be $d!$ times the Euclidean volume of $\text{conv}(A \cup \{0\})$ the convex hull of the columns of A and the origin in \mathbb{R}^d . The following theorem gives the parameters β for which $\text{rank}(H_A(\beta))$ is higher than expected:

Theorem 3.1. [MMW05] *Let $H_A(\beta)$ be an A -hypergeometric system with parameter β . If $\beta \in \text{qdeg}(\bigoplus_{i=0}^{d-1} H_{\mathfrak{m}}^i(R/I_A))$ then $\text{rank}(H_A(\beta)) > \text{vol}(A)$. Otherwise, $\text{rank}(H_A(\beta)) = \text{vol}(A)$.*

Since Theorem 3.1 was the initial motivation for `Quasidegrees`, the package has a method `quasidegreesLocalCohomology` (abbreviated `QLC`) to compute the quasidegree set of the local cohomology modules $H_{\mathfrak{m}}^i(R/I_A)$. If the input is an integer i and the R -module R/I_A , then the method computes $\text{qdeg}(H_{\mathfrak{m}}^i(R/I_A))$. If the input is only the module R/I_A , the method computes the quasidegree set in Theorem 3.1.

We use graded local duality to compute the local cohomology modules of a finitely generated A -graded R -module supported at the maximal ideal \mathfrak{m} :

Theorem 3.2. (Graded local duality [BH98, Mil01]) *Given an A -graded R -module M , there is an A -graded vector space isomorphism*

$$\text{Ext}_R^{n-i}(M, R)_{\alpha} \cong \text{Hom}_{\mathbb{k}}(H_{\mathfrak{m}}^i(M)_{-\alpha-\varepsilon_A}, \mathbb{k})$$

where $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$ and $\varepsilon_A = \sum_{j=1}^n a_j$.

The algorithm implemented for `quasidegreesLocalCohomology` is essentially Algorithm 1 applied to the `Ext`-modules of M with the additional twist of ε_A coming from local duality. For our purposes, we exploit the fact that the higher syzygies of R/I_A are generated by monomials in R^m (see [MS05], Chapter 9).

Continuing our running example, we use `quasidegreesLocalCohomology` to compute the quasidegree set of $\bigoplus_{i=0}^{d-1} H_{\mathfrak{m}}^i(R/I_A)$:

```
i11 : M=R^1/I
o11 = cokernel | x_1x_3-x_2x_4 x_1x_4^2-x_3^2x_5
x_1^2x_4-x_2x_3x_5 x_1^3-x_2^2x_5 |
1
o11 : R-module, quotient of R
i12 : quasidegreesLocalCohomology M
o12 = {{| 0 |, {| 1 |}}
      | 0 |   | 0 |
      | 1 |   | -2 |
o12 : List
```

Thus

$$(1) \quad \text{qdeg} \left(\bigoplus_{i=0}^{d-1} H_{\mathfrak{m}}^i(R/I_A) \right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \mathbb{C} \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}.$$

As a check, we use the methods `gkz` and `holonomicRank` from the package `Dmodules` to compute $\text{rank}(H_A(0))$ and $\text{rank}(H_A(\beta))$ for two different β in (1) and demonstrate a rank jump:

```
i13 : holonomicRank gkz(A,{0,0,0}) -- vol A in this case
o13 = 4
i14 : holonomicRank gkz(A,{0,0,1})
o14 = 5
i15 : holonomicRank gkz(A,{3/2,0,-2})
o15 = 5
```

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